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Orthomodular Bell-Kochen-Specker Theorem

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Four elements in an orthomodular lattice of height four generate a partial Boolean subalgebra that contains a Bell-Kochen-Specker theorem. This result directly explains and generalizes the 4-dimensional Bell-Kochen-Specker theorems of various authors.

KEY WORDS: orthomodular lattice; Bell-Kochen-Specker theorem; partial Boolean algebra.

1. INTRODUCTION

In Smith (1999), it is shown that any 4-generator partial Boolean algebra (pBa) induced by an orthomodular lattice of height 4 or less is finite, establishing that the set of 5 generators for Conway and Kochen's infinite pBa in $\mathbb{C}^4$ (Kochen, 1996) is minimal in number. Here, we show that one of the 4-generator cases, when the generators a, b, c, d have only the compatibilities $a \diamond b \diamond c \diamond d \diamond a$, leads to a Bell-Kochen-Specker (BKS) theorem which has a purely orthomodular character. This example explains and generalizes the 4-dimensional BKS theorems of many authors, including Peres (1990, 1993), Mermin (1990, 1993), Kernaghan (1994) and Cabello *et al.* (1996a,b).

Recent papers have attempted to minimize the size of a configuration of vectors in $\mathbb{R}^4$, as well as the number of equations involving those vectors, while maintaining a contradiction with the sum rules of quantum mechanics. Another, and possibly more meaningful, measure of the simplicity of a BKS configuration is the number of elements required to generate a pBa containing a BKS theorem. The example we present shows that the minimum for this measure is four, since Coray (1970) has shown that any 3-generator pBa can be embedded in a Boolean algebra.

2. A 4-GENERATOR PARTIAL BOOLEAN ALGEBRA

We follow the notations of Kalmbach (1983) in discussing orthomodular lattices, except that we use $\diamond$ to denote compatibility. Within the orthomodular

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lattice L described below, we will be interested in the structure of $B(a, b, c, d)$, the pBa generated by a, b, c, d. See (Kochen and Specker, 1967; Kochen, 1996) for discussions of partial Boolean algebras.

We set $L = \Gamma\{a, b, c, d\}$ to be an orthomodular lattice generated by distinct, non-trivial, and pairwise non-complementary elements a, b, c, d with $a \diamond b \diamond c \diamond d \diamond a$. We wish to keep the height of L as small as possible, so we insist that all of the (central) elements $a^{\epsilon_a} \wedge c^{\epsilon_c}$ and $b^{\epsilon_b} \wedge d^{\epsilon_d}$ are equal to 0, where each ϵ_α determines either α or its orthocomplement α', so that

$$MO2 \cong \Gamma\{a, c\} \cong \Gamma\{b, d\}.$$

This implies that we must have, for example, $d \not\leq (a^{\epsilon_a} \wedge b^{\epsilon_b})$, since otherwise we find that $a^{\epsilon_a} = 0$. So each $a^{\epsilon_a} \wedge b^{\epsilon_b} \neq 0$, and thus

$$\mathbf{2}^4 \cong \Gamma\{a, b\} \cong \Gamma\{b, c\} \cong \Gamma\{c, d\} \cong \Gamma\{d, a\},$$

the latter three isomorphisms following from the first by the cyclic symmetry of the compatibilities among the generators.

Define

$$\begin{aligned} x &= (a \wedge b) \vee (a' \wedge b') \\ y &= (b \wedge c) \vee (b' \wedge c') \\ z &= (c \wedge d) \vee (c' \wedge d') \\ w &= (d \wedge a) \vee (d' \wedge a'). \end{aligned}$$

Each of these elements lies on the middle level of the respective $\mathbf{2}^4$ displayed above and thus is an element of a generating pair for this Boolean algebra. Figure 1 displays the compatibilities among the eight named elements by edges joining them, with any two elements on a cycle of three edges (thought of as on a flat torus) generating a $\mathbf{2}^4$.

In the case that $x \diamond z$ and $w \diamond y$, we see that

$$L \cong \Gamma\{a, x, z, d\} \cong \Gamma\{b, x, z, c\} \cong \Gamma\{a, b, y, w\} \cong \Gamma\{d, c, y, w\},$$

with the same cyclic compatibilities among the four new sets generating L as for a, b, c, d. Thus the above arguments for a, b, c, d may equally be applied to them. In particular, we have

$$\mathbf{2}^4 \cong \Gamma\{x, z\} \cong \Gamma\{y, w\},$$

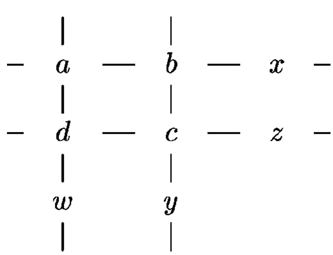

Fig. 1. Compatibilities among certain generators of $\mathbf{2}^4$ subalgebras in L.

and we set

$$u = (x \wedge z) \vee (x' \wedge z')$$
$$v = (y \wedge w) \vee (y' \wedge w').$$

What's striking is that we must now have $u' = v$. For on the one hand,

$$\begin{aligned} y \wedge w &= ((b' \vee c) \wedge (b \vee c')) \wedge ((a' \vee d) \wedge (a \vee d')) \\ &= ((b' \vee c) \wedge (a' \vee d)) \wedge ((b \vee c') \wedge (a \vee d')) \\ &= ((a' \wedge b') \vee (c \wedge d)) \wedge ((a \wedge b) \vee (c' \wedge d')) \\ &\leq ((a' \wedge b') \vee (c \wedge d)) \vee ((a \wedge b) \vee (c' \wedge d')) \\ &= ((a \wedge b) \vee (a' \wedge b')) \vee ((c \wedge d) \vee (c' \wedge d')) \\ &= x \vee z, \end{aligned}$$

where the third equality uses the fact that $\{a', b', c, d\}$ and $\{a, b, c', d'\}$ are Greechie sets. On the other hand,

$$\begin{aligned} y \wedge w &= ((b' \vee c) \wedge (b \vee c')) \wedge ((a' \vee d) \wedge (a \vee d')) \\ &= ((b' \vee c) \wedge (a \vee d')) \wedge ((b \vee c') \wedge (a' \vee d)) \\ &= ((a \wedge b') \vee (c \wedge d')) \wedge ((a' \wedge b) \vee (c' \wedge d)) \\ &\leq ((a \wedge b') \vee (c \wedge d')) \vee ((a' \wedge b) \vee (c' \wedge d)) \\ &= ((a' \wedge b) \vee (a \wedge b')) \vee ((c \wedge d') \vee (c' \wedge d)) \\ &= x' \vee z'. \end{aligned}$$

Thus, $y \wedge w \leq ((x \vee z) \wedge (x' \vee z')) = u'$. By a similar computation, $y' \wedge w' \leq u'$, and thus $v = ((y \wedge w) \vee (y' \wedge w')) \leq u'$.

Writing $v = ((y' \vee w) \wedge (y \vee w'))$ and $u' = ((x' \wedge z) \vee (x \wedge z'))$, a similar computation shows that $u' \leq v$, and so we conclude that $u' = v$.

Figure 2 displays the compatibilities among the ten named elements. The pBa $B(a, b, c, d)$ is then comprised of precisely the elements contained in any of the six copies of $\mathbf{2}^4$.

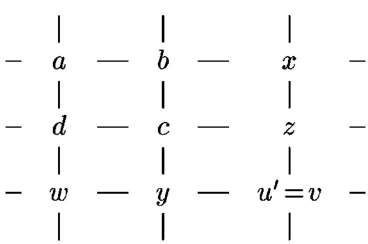

Fig. 2. Compatibilities within all $\mathbf{2}^4$ subalgebras in L.

3. EXPLAINING 4-DIMENSIONAL BKS THEOREMS

A BKS theorem can be seen in Fig. 2 in the following way. Each of the six 3-cycles in Fig. 2 corresponds to a Boolean subalgebra $\mathbf{2}^4$ with four atoms. Seeing that each of the sets

$$\{a, b, x\}, \{b, c, y\}, \{c, d, z\}, \{d, a, w\}, \{x, z, u\}, \{y, w, v\}$$

has the property that each of its elements covers a common atom, a 0-1 function assigning 1 to one atom and 0 to the other three atoms in any $\mathbf{2}^4$ will cause 1 to be assigned to an atom covered by either one or three elements in each set above. But the last column of Fig. 2 displays $\{x, z, u'\}$, not $\{x, z, u\}$, in which either zero or two of its elements will cover the atom assigned the value 1. A contradiction now comes from the fact that the sum over the rows of Fig. 2 is odd, while over the columns it is even.

Figure 3 presents a rational example of this phenomenon in $\mathbb{R}^4$. The BKS theorems in $\mathbb{R}^4$ of Peres, Kerhaghan, and Cabello *et al.* rely on vectors determined by the 2-dimensional subspaces in Fig. 3. Figure 3 may be seen as depicting the compatibilities among the nine 2-dimensional subspaces spanned by pairs of vectors taken from the following six tetrads of orthogonal vectors in $\mathbb{R}^4$:

	v_0	v_1	v_2	v_3
$C_1 =$	++00	+−00	00++	00+−
$C_2 =$	+0+0	+0−0	0+0+	0−0+
$C_3 =$	+00+	+00−	0++0	0+−0
$R_1 =$	+000	0+00	00+0	000+
$R_2 =$	++++	++−	+−+−	+−−+
$R_3 =$	+++−	++−+	+−++	−+++

Here, + represents $+1$ and − represents -1. The $(i, j)th$ entry of Fig. 3 is both the 2-space spanned by v_0 and v_i of "column" C_j and the 2-space spanned by v_0 and v_j of "row" R_i, except when $i = j = 3$. In that case, the 2-spaces are orthogonal complements.

$a =$ ++00 +−00	$b =$ +0+0 +0−0	$x =$ +00+ +00−
$d =$ ++00 00++	$c =$ +0+0 0+0+	$z =$ +00+ 0++0
$w =$ ++00 00+−	$y =$ +0+0 0+0−	$u' =$ +00− 0++0

Fig. 3. A rational 4-generator Bell-Kochen-Specker theorem in $\mathbb{R}^4$.

The BKS theorem of Peres is based on the fact that no appropriate 0-1 function exists for the six tetrads above. Kernaghan and Cabello *et al.*, also use a subset of the 1-dimensional subspaces corresponding to the 24 vectors displayed above. It is not surprising that these vectors (and their negatives) have been utilized, since properly scaled they correspond to the vectors of norm 1 and 2 in the integral lattice D_4, one of the two most useful 4-dimensional lattices (Conway and Sloane, 1993; pp. 118–119). The Voronoi cell of this lattice is the regular 4-dimensional polytope called the 24-cell (Coxeter, 1973), which has been mentioned in Peres (1993) and Aravind and Lee-Elkin (1998), the latter reference also describing BKS theorems based on the two dual 4-dimensional polytopes known as the 120- and 600-cells. Among other uses, the lattice D_4 also corresponds to the lattice of Hurwitz integral quaternions (Conway and Smith, 2003).

Mermin, following Peres, presents a nine-element Kochen-Specker contradiction in $\mathbb{C}^4$. This multiplicative example corresponds to Fig. 2 by setting $a = \sigma_x \otimes 1$, $b = 1 \otimes \sigma_x$, $c = \sigma_y \otimes 1$, and $d = 1 \otimes \sigma_y$, where σ_x and σ_y are Pauli spin matrices.

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